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GEOMETRY.

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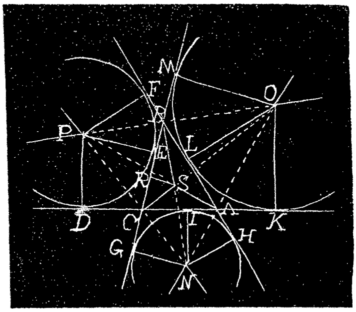
97. Proposed by CHARLES C. CROSS, Libertytown, Md.

Prove by pure geometry: The radius of a circle drawn through the centers of the inscribed and any two escribed circles of a triangle is double the radius of the circumscribed circle of the triangle.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.



Let ABC=any triangle, S be the center of the inscribed circle, and P, N, and O, the centers of the escribed circles. Put $BC=a$, $AC=b$, $AB=c$, $SR=r$, $PE=r_1$, $NH=r_2$, $OM=r_3$, $\dfrac{a+b+c}{2}=s$, $\triangle$=area of $\triangle ABC$.

$$\triangle ABC=\triangle BCS+\triangle ACS+\triangle ABS$$

$$=\frac{ra}{2}+\frac{rb}{2}+\frac{rc}{2}=\tfrac{1}{2}r(a+b+c)=rs \; \ldots\ldots \; (1).$$

Polygon $ACPB=\triangle ABC=\triangle CPB=\triangle ACP+\triangle ABP$; or $rs+\dfrac{r_1a}{2}=\dfrac{r_1b}{2}=\dfrac{r_1c}{2}$; whence $rs=r_1\left(\dfrac{a+b+c}{2}-a\right)$, or $rs=r_1(s-a)$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (2).

Polygon $BANC=\triangle ABC+\triangle ANC=\triangle BAN+\triangle BCN$; or $rs+\dfrac{r_2b}{2}=\dfrac{r_2c}{2}=\dfrac{r_2a}{2}$; whence $rs=r_2\left(\dfrac{a+b+c}{2}-b\right)$, or $rs=r_2(s-b)$ $\ldots\ldots\ldots\ldots\ldots\ldots$ (3).

Similarly, we find $rs=r_3\left(\dfrac{a+b+c}{2}-c\right)$, or $rs=r_3(s-c)$ $\ldots\ldots\ldots\ldots$ (4).

Then (1), (2), (3), and (4), respectively, give $rs=\triangle$, $r_1(s-a)=\triangle$, $r_2(s-b)=\triangle$, and $r_3(s-c)=\triangle$. Whence $r=\dfrac{\triangle}{s}$, $r_1=\dfrac{\triangle}{s-a}$, $r_2=\dfrac{\triangle}{s-b}$, and $r_3=\dfrac{\triangle}{s-c}$.

As it is readily proved, we simply state that the lines drawn from the centers of the escribed circles, respectively, to the vertices of the opposite angles of the triangle, are (1) The bisectors respectively of the angles of the triangle, and hence pass through the center of the inscribed circle, and (2) The perpendiculars respectively to the lines joining the centers of the escribed circles; as BN bisects $\angle ABC$ and is perpendicular to PO.

Again, leaving the burden of proof to the reader, hinting, however, that $\angle MBL$ is bisected by PO as a property of escribed circles, we find the following sets of triangles similar: PEB and OLB; OLA and NIA; NIC and PEC.

Let $EC=x$, $IC=y$, $AL=z$; then $BE=a-x$, $AI=b-y$, $BL=c-z$.

From the sets of similar triangles just noted we obtain the following proportions, remembering that $PE=r_1$, $NI=r_2$, and $OL=r_3$.

$c-z : r_3=a-x : r_1$, or $r_1(c-z)=r_3(a-x)$ ....(1).

$z : r_3=b-y : r_2$, or $z=\dfrac{r_3(b-y)}{r_2}$ ....(2).

$y : r_2=x : r_1$, or $x=\dfrac{r_1y}{r_2}$ ....(3).

Substituting the values of z and x, of (2) and (3), in (1), we find

$$y=\tfrac{1}{2}b+\frac{r_2a}{2r_1}-\frac{r_2c}{2r_3}=\tfrac{1}{2}b+\frac{a(s-a)}{2(s-b)}-\frac{c(s-c)}{2(s-b)}=s-a.$$

Whence $b-y=s-c$, $x=s-b$, $a-x=s-c$, $z=s-b$, and $c-z=s-a$.

Then $PB=\sqrt{[(a-x)^2+r_1^2]}=\sqrt{\left(\dfrac{ac(s-c)}{s-a}\right)}$; $BO=\sqrt{\left(\dfrac{ac(s-a)}{s-c}\right)}$; $PO=PB+BO=\dfrac{b\sqrt{(ac)}}{\sqrt{[(s-a)(s-c)]}}$.

Also $AO=\sqrt{\left(\dfrac{bc(s-b)}{s-c}\right)}$; $AN=\sqrt{\left(\dfrac{bc(s-c)}{s-b}\right)}$; $NO=AO+AN=\dfrac{a\sqrt{(bc)}}{\sqrt{[(s-b)(s-c)]}}$.

Also $NC=\sqrt{\left(\dfrac{ab(s-a)}{s-b}\right)}$; $PC=\sqrt{\left(\dfrac{ab(s-b)}{s-a}\right)}$; $NP=NC+PC=\dfrac{c\sqrt{(ab)}}{\sqrt{[(s-a)(s-b)]}}$.

Also $NB=\sqrt{\left(\dfrac{acs}{s-b}\right)}$; $PA=\sqrt{\left(\dfrac{bcs}{s-a}\right)}$; $OC=\sqrt{\left(\dfrac{abs}{s-c}\right)}$

From the similar triangles PAN and PCS, we have

$PN : PA=PS : PC$, or $\dfrac{c\sqrt{(ab)}}{\sqrt{[(s-a)(s-b)]}} : \sqrt{\left(\dfrac{bcs}{s-a}\right)}=PS : \sqrt{\left(\dfrac{ab(s-b)}{s-a}\right)}$;

whence $PS=\dfrac{a\sqrt{(bc)}}{\sqrt{[s(s-a)]}}$.

Similarly, $OS=\dfrac{c\sqrt{(ab)}}{\sqrt{[s(s-c)]}}$, and $NS=\dfrac{b\sqrt{(ac)}}{\sqrt{[s(s-b)]}}$.

The radius of the circumscribed circle of any triangle equals the product of the three sides divided by four times the area of the triangle.

Put the radius of the circumscribed circle of $\triangle ABC=R$, of $\triangle NSO=R_1$, of $\triangle PSO=R_2$, of $\triangle PSN=R_3$, and of $\triangle PON=R_4$.

Sides of $\triangle ABC$ are a, b, c; and area$=\sqrt{[s(s-a)(s-b)(s-c)]}=\triangle$.

$$\therefore R=\frac{abc}{4\triangle}.$$

Sides of $\triangle NSO$ are $\dfrac{a\sqrt{(bc)}}{\sqrt{[(s-b)(s-c)]}}$, $\dfrac{c\sqrt{(ab)}}{\sqrt{[s(s-c)]}}$, $\dfrac{b\sqrt{(ac)}}{\sqrt{[s(s-b)]}}$; and area$=\dfrac{abc\sqrt{(s-a)}}{2\sqrt{[s(s-b)(s-c)]}}$.

$$\therefore R_1=\frac{abc}{2\triangle}=2R:$$

Sides of $\triangle PSO$ are $\dfrac{b\surd(ac)}{\surd[(s-a)(s-c)]}$, $\dfrac{c\surd(ab)}{\surd[s(s-c)]}$, $\dfrac{a\surd(bc)}{\surd[s(s-a)]}$; and area $=\dfrac{abc\surd(s-b)}{2\surd[s(s-a)(s-c)]}$.

$$\therefore R_2=\frac{abc}{2\triangle}=2R.$$

Sides of $\triangle PSN$ are $\dfrac{c\surd(ab)}{\surd[(s-a)(s-b)]}$, $\dfrac{a\surd(bc)}{\surd[s(s-a)]}$, $\dfrac{b\surd(ac)}{\surd[s(s-b)]}$; and area $=\dfrac{abc\surd(s-c)}{2\surd[s(s-a)(s-b)]}$.

$$\therefore R_3=\frac{abc}{2\triangle}=2R.$$

Sides of $\triangle PON$ are $\dfrac{b\surd(ac)}{\surd[(s-a)(s-c)]}$, $\dfrac{a\surd(bc)}{\surd[(s-b)(s-c)]}$, $\dfrac{c\surd(ab)}{\surd[(s-a)(s-b)]}$, and area $=\dfrac{abc\surd s}{2\surd[(s-a)(s-b)(s-c)]}$.

$$\therefore R_4=\frac{abc}{2\triangle}=2R.$$

Hence the general proposition : The centers of the escribed and inscribed circles of a triangle are the vertices of four different triangles the radii of whose circumscribed circles are equal.

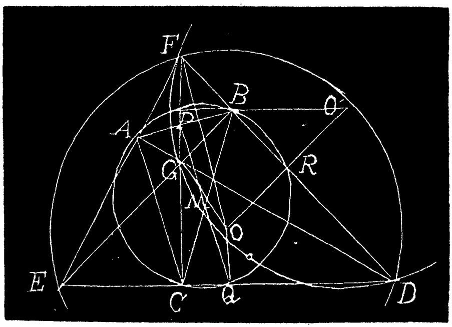
Take $a=14$, $b=13$, $c=15$. Then $r=4$, $r_1=12$, $r_2=10\frac{1}{2}$, $r_3=14$, and $R=8\frac{1}{8}$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let ABC be the given triangle, then if G is the incenter of ABC, E, F, D are its excenters. The circumcircle of ABC is the nine-points circle of EDF and G its orthocenter. Hence PQ is a diameter of the nine-points circle, since P is the mid-point of FG and Q the mid-point of ED.

$\therefore$ The mid-point of PQ is the center of the nine-points circle.

Let O be the circumcenter. Then the mid-point of GO is the center of the nine-points circle ; since perpendiculars to the mid-points of CQ and BR, chords of the nine-points circle, intersect at the mid-point of GO and also at the center of the nine-points circle.



$\therefore$ PQ and GO bisect one another at M. From the triangles MQO and GPM, $GM=OM$, $PM=QM$, $\angle OMQ=\angle PMG$. $\therefore$ $OQ=PG=FP$, and OQ is also parallel to FP. $\therefore$ $OF=PQ$.

$\therefore$ The radius of EFD=the diameter of ABC. Perpendicular to FG at the point P draw $PO'=QD$. Then since $FP=PG=OQ$, $O'F=O'G=OD$.

$\therefore$ $OF=O'F=OD=O'D=O'G$.

$\therefore$ radius of FGD=radius EFD=diameter ABC.

Similarly radius EGD and radius EGF=radius (each) EFD.

98. Proposed by **EDWARD R. ROBBINS, Master in Mathematics and Physics, Lawrenceville School, Lawrenceville, N. J.**

Construct a circle which shall pass through two given points and touch a given circle, (1) when the distance between the points is less than the diameter of the circle, and (2) when it is greater.

I. Solution by **G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

The construction for both cases is the same.

I Case. Let A, B be the given points, $CMDK$ the given circle. Through A, B draw the circle $ABEF$ intersecting the given circle in E, F. Draw AB, EF intersecting at G. Draw the tangents GK, GM. Draw QQ_1 perpendicular to AB at its mid-point. Through R, the center of $CMDK$, draw RM, RK intersecting QQ_1 in O_1, O. Then O, O_1 are the centers of two circles satisfying the conditions, and ABK, ABM are the circles.

II Case. Let C, D be the given points, $AHBL$ the given circle. Through C, D describe the circle $CDEF$ intersecting the given circle in E, F. Draw CD, EF intersecting at G. Draw the tangents GH, GL. Draw RR_1 perpendicular to CD at its mid-point. Through Q, the center of $AHBL$, draw QL, QH intersecting RR_1 in P, P_1. Then P, P_1 are the centers of two circles satisfying the conditions, and CDL, CDH are the circles.

In the above both points are without the given circle. This problem is thoroughly discussed on page 271, No. 8, Vol. I., THE AMERICAN MATHEMATICAL MONTHLY.

II. Solution by **FREDERIC R. HONEY, Ph. B., New Haven, Conn.**

The following description applies when the distance between the points is less, and when greater than the diameter of the circle.

Let a and b be the given points and A the circumference of the given circle.

Through a and b pass a circle the circumference of which intersects A at c and d. Draw ba and dc and produce these lines until they meet at e. Draw ef tangent to A. Through the point of tangency f and the given points a and b pass the required circle C. Since two tangents may be drawn there are, in each case, two solutions.

Analysis of the construction: $eb\times ea=ed\times ec=(ef)^2$.

[NOTE. For a demonstration of this same proposition with a diagram, see Vol. I., page 271. Professors Zerr and Honey each furnished neat diagrams with these demonstrations, but we believe the demonstrations sufficiently clear without them. ED. F.]